

REGULARITY OF CANONICAL AND DEFICIENCY MODULES FOR MONOMIAL IDEALS

MANOJ KUMMINI AND SATOSHI MURAI

ABSTRACT. We show that the Castelnuovo–Mumford regularity of the canonical or a deficiency module of the quotient of a polynomial ring by a monomial ideal is bounded by its dimension.

1. INTRODUCTION

Let $R = \mathbb{k}[x_1, \dots, x_n]$ be a standard graded polynomial ring over a field \mathbb{k} and $\mathfrak{m} = (x_1, \dots, x_n)$ the homogeneous maximal ideal of R . In this paper, we study the Castelnuovo–Mumford regularity of the modules $\text{Ext}_R^i(R/I, \omega_R)$ when $I \subset R$ is a monomial ideal; here $\omega_R = R(-n)$ denotes the canonical module of R . The $\text{Ext}_R^i(R/I, \omega_R)$, $i > n - \dim R/I$ are called *deficiency modules* of R/I while $\text{Ext}_R^{n-\dim R/I}(R/I, \omega_R)$ is called the *canonical module* of R/I .

For any homogeneous ideal $I \subseteq R$, local cohomology modules $H_{\mathfrak{m}}^i(R/I)$ are important in commutative algebra and algebraic geometry. One is often interested in the vanishing of homogeneous components of $H_{\mathfrak{m}}^i(R/I)$. While one cannot expect the vanishing of $H_{\mathfrak{m}}^i(R/I)$ in negative degrees, unless it has finite length, one can, using the local duality theorem of Grothendieck, obtain some information from $\text{Ext}_R^{n-i}(R/I, \omega_R)$. For a finitely generated graded R -module M , its (*Castelnuovo–Mumford*) *regularity*, $\text{reg}(M)$, is an invariant that contains information about the stability of homogeneous components in sufficiently large degrees. In light of these, it is desirable to get bounds on $\text{reg}(\text{Ext}_R^i(R/I, \omega_R))$. Such bounds were studied by L. T. Hoa and E. Hyry [HH06] and M. Chardin, D. T. Ha and Hoa [CHH09]; see also the references in those papers.

Unfortunately, canonical and deficiency modules can have large regularity. For a finitely generated graded R -module M , known bounds for $\text{reg}(\text{Ext}_R^i(M, \omega_R))$ are large (see, *e.g.*, [HH06, Theorems 9 and 14]). On the other hand, more optimal bounds for $\text{reg}(\text{Ext}_R^i(R/I, \omega_R))$ are known to exist for certain classes of graded ideals I (see [HH06, Section 4]). It is an interesting problem to find a class of graded ideals $I \subset R$ with optimal bounds for $\text{reg}(\text{Ext}_R^i(R/I, \omega_R))$. In this paper, we focus on monomial ideals. It follows from the theory of square-free modules, introduced by K. Yanagawa [Yan00], that if I is a square-free monomial ideal then $\text{reg}(\text{Ext}_R^i(R/I, \omega_R)) \leq \dim \text{Ext}_R^i(R/I, \omega_R)$. This bound is small, since $\dim \text{Ext}_R^i(R/I, \omega_R) \leq n - i$ (see [BH93, Corollary 3.5.11]).

While one cannot apply the theory of square-free modules to all monomial ideals, there are results that show that, when I is a monomial ideal, $\text{reg}(\text{Ext}_R^i(R/I, \omega_R))$ is not large. For example, we see from [Tak05, Proposition 1, p 333] that if $\text{Ext}_R^i(R/I, \omega_R)$ has finite length then its regularity is negative or equal to zero.

Again, Hoa and Hyry [HH06, Proposition 21] showed that if $H_m^i(R/I)$ has finite length for $i = 0, 1, \dots, d-1$, where $d = \dim R/I$, then $\text{reg}(\text{Ext}_R^{n-d}(R/I, \omega_R)) \leq d$. We generalize these results in the following theorem:

Theorem 1.1. *Let $I \subseteq R$ be a monomial ideal. Then, for all $0 \leq i \leq n$,*

$$\text{reg}(\text{Ext}_R^i(R/I, \omega_R)) \leq \dim \text{Ext}_R^i(R/I, \omega_R).$$

Since $\dim \text{Ext}_R^i(R/I, \omega_R) \leq n - i$ we immediately get:

Corollary 1.2. *Let $I \subseteq R$ be a monomial ideal. Then, for all $0 \leq i \leq n$,*

$$\text{reg}(\text{Ext}_R^i(R/I, \omega_R)) \leq n - i.$$

The above conclusion need not hold, in general, without the assumption that I is a monomial ideal; see [CD03, Example 3.5].

Our approach to bounding the regularity of canonical and deficiency modules differs from that of Hoa and Hyry. We show that if I is a monomial ideal, then $\text{Ext}_R^i(R/I, \omega_R)$ has a multigraded filtration, called *Stanley filtration*, introduced by D. Maclagan and G. G. Smith [MS05]; the bound on regularity follows from this filtration.

In the next section, we discuss some preliminaries on Stanley filtrations and local cohomology. In Section 3 we prove our main result.

2. PRELIMINARIES

Hereafter we take R -modules to be graded by \mathbb{Z}^n , giving $\deg x_i = \mathbf{e}_i$, the i th unit vector of \mathbb{Z}^n . We call this the *multigrading* of R and R -modules.

Notation 2.1. Let $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{Z}^n$. Write $\mathbf{x}^{\mathbf{a}} = \prod_{i=1}^n x_i^{a_i} \in \mathbb{k}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. We say that \mathbf{a} is the *degree* of $\mathbf{x}^{\mathbf{a}}$, and write $\deg \mathbf{x}^{\mathbf{a}} = \mathbf{a}$. Define $\text{Supp}(\mathbf{a}) = \{i : a_i \neq 0\}$. Define $\mathbf{a}^+, \mathbf{a}^- \in \mathbb{N}^n$ by the conditions $\mathbf{a} = \mathbf{a}^+ - \mathbf{a}^-$ and $\text{Supp}(\mathbf{a}^+) \cap \text{Supp}(\mathbf{a}^-) = \emptyset$. We write $\|\mathbf{a}\|$ for $\sum_{i=1}^n a_i$, the *total degree* of \mathbf{a} (and of the monomial $\mathbf{x}^{\mathbf{a}}$). We will say that \mathbf{a} (equivalently $\mathbf{x}^{\mathbf{a}}$) is *square-free* if $a_i \in \{0, 1\}$ for all i . Let $[n] = \{1, \dots, n\}$. For $\Lambda \subseteq [n]$, we set $\mathbf{e}_\Lambda = \sum_{i \in \Lambda} \mathbf{e}_i$ and abbreviate the (square-free) monomial $\mathbf{x}^{\mathbf{e}_\Lambda}$ as x_Λ . The canonical module of R is $\omega_R = R(-\mathbf{e}_{[n]})$.

Let M be a finitely generated multigraded R -module. Let $m \in M$ be a homogeneous element and let $G \subset \{x_1, \dots, x_n\}$ be a subset such that $um \neq 0$ for all monomials $u \in \mathbb{k}[G]$. The \mathbb{k} -subspace $\mathbb{k}[G]m$ of M generated by all the um , where u is a monomial in $\mathbb{k}[G]$, is called a *Stanley space*. A *Stanley decomposition* of M is a finite set \mathcal{S} of pairs (m, G) of homogeneous elements $m \in M$ and $G \subseteq \{x_1, \dots, x_n\}$ such that $\mathbb{k}[G]m$ is a Stanley space for all $(m, G) \in \mathcal{S}$ and

$$(1) \quad M =_{\mathbb{k}} \bigoplus_{(m, G) \in \mathcal{S}} \mathbb{k}[G]m$$

(We use $=_{\mathbb{k}}$ to emphasize that the decomposition is only as vector spaces.) Properties of such decompositions have been widely studied; we follow the approach of [MS05, Section 3] where Stanley decompositions were used to get bounds for multigraded regularity. Following [MS05, Definition 3.7], we define a *Stanley filtration* to be a Stanley decomposition with an ordering of pairs $\{(m_i, G_i) : 1 \leq i \leq p\}$

such that, for $j = 1, 2, \dots, p$,

$$\left(\sum_{i=1}^j Rm_i \right) / \left(\sum_{i=1}^{j-1} Rm_i \right) = \mathbb{k}[G_j](-\deg m_j).$$

as R -modules. Note, in this case, that

$$0 \subseteq Rm_1 \subseteq \dots \subseteq \sum_{i=1}^j Rm_i \subseteq \dots \subseteq \sum_{i=1}^p Rm_i = M$$

is a prime filtration of M , as in [Eis95, p. 93, Proposition 3.7].

Proposition 2.2. *Let M be a multigraded R -module with a Stanley decomposition \mathcal{S} such that for all $(m, G) \in \mathcal{S}$, $(\deg m)^+$ is square-free and $G = \text{Supp}((\deg m)^+)$. Then \mathcal{S} gives a Stanley filtration. Moreover $\text{reg } M \leq \max\{\|\deg m\| : (m, G) \in \mathcal{S}\}$.*

Proof. We order $\mathcal{S} = \{(m_1, G_1), \dots, (m_p, G_p)\}$ so that $\|\deg m_1\| \geq \dots \geq \|\deg m_p\|$. It follows from our hypothesis that

$$(2) \quad \text{span}_{\mathbb{k}}\{m_1, \dots, m_p\} = \text{span}_{\mathbb{k}}\{m \in M : \text{Supp}((\deg m)^+) \text{ is square-free}\},$$

where $\text{span}_{\mathbb{k}}(V)$ denotes the \mathbb{k} -vector space spanned by elements in V . Write $M^{(j)}$ for $\sum_{i=1}^j Rm_i$. We will now show, inductively on j , that

$$(A) \quad M^{(j-1)} :_R m_j = (x_k; x_k \notin G_j).$$

$$(B) \quad \text{The set } \cup_{i=1}^j \{um_i : u \text{ is a monomial in } \mathbb{k}[G_i]\} \text{ is a } \mathbb{k}\text{-basis for } M^{(j)}.$$

They imply that \mathcal{S} is a Stanley filtration of M .

Let $j = 1$. We will show $(0 :_R m_1) = (x_k; x_k \notin G_1)$. For all monomials $u \in \mathbb{k}[G_1]$, $um_1 \neq 0$, from the definition of the decomposition. Therefore we must show that $x_l m_1 = 0$ for any $x_l \notin G_1$. Let $x_l \notin G_1$. Then $(\deg x_l m_1)^+$ is square-free, and by (2), $x_l m_1 \in \text{span}_{\mathbb{k}}\{m_1, \dots, m_p\}$. However, from the choice of m_1 , we see that $x_l m_1 = 0$. Therefore $(0 :_R m_1) = (x_k; k \notin G_1)$ proving (A). Note that (B) follows immediately.

Now assume that $j > 1$ and that the assertion is known for all $i < j$. We first show (A). Let u be a monomial in $\mathbb{k}[G_j]$. By the statement (B) for $j - 1$, the set $\cup_{i=1}^{j-1} \{vm_i : v \text{ is a monomial in } \mathbb{k}[G_i]\}$ is a \mathbb{k} -basis for $M^{(j-1)}$. Since um_j is an element of the basis of M coming from the Stanley decomposition, um_j is not in the \mathbb{k} -linear span of $\cup_{i=1}^{j-1} \{vm_i : v \text{ is a monomial in } \mathbb{k}[G_i]\}$, i.e., $um_j \notin M^{(j-1)}$. It remains to prove that $x_l m_j \in M^{(j-1)}$ for any $x_l \notin G_j$. Let $x_l \notin G_j$. Since $(\deg x_l m_j)^+$ is square-free, it follows, from (2) and the ordering of the (m_i, G_i) , that

$$x_l m_j \in \text{span}_{\mathbb{k}}\{m_i : 1 \leq i \leq p, \deg m_i > \deg m_j\} \subseteq \text{span}_{\mathbb{k}}\{m_1, \dots, m_{j-1}\}.$$

Therefore $x_l m_j \in M^{(j-1)}$, proving the statement (A) for j .

From (A), we see that the following sequence is exact:

$$(3) \quad 0 \longrightarrow M^{(j-1)} \longrightarrow M^{(j)} \longrightarrow \mathbb{k}[G_j]m_j \longrightarrow 0.$$

Now statement (B) for j follows from the induction hypothesis.

The assertion about regularity is essentially [MS05, Theorem 4.1], but we give a quick proof here. We will show that $\text{reg } M^{(j)} \leq \max\{\|\deg m_i\| : 1 \leq i \leq j\}$ for all $1 \leq j \leq p$. It holds for $j = 1$. For $j > 1$, it follows from [Eis95, Corollary 20.19] and the exact sequence (3) that $\text{reg } M^{(j)} \leq \max\{\text{reg } M^{(j-1)}, \|\deg m_j\|\}$; induction completes the proof. \square

Finally, we recall some basics of local cohomology, following [BH93, Sections 3.5–3.6]. Let \check{C}^\bullet be the Čech complex on x_1, \dots, x_n ; the term at the i th cohomological degree is

$$\check{C}^i = \bigoplus_{\Lambda \subseteq [n], |\Lambda|=i} R_{x_\Lambda}$$

where R_{x_Λ} denotes inverting the monomial x_Λ . Note that \check{C}^\bullet is a complex of \mathbb{Z}^n -graded R -modules, with differentials of degree 0. For a finitely generated R -module M , we set $\check{C}^\bullet(M) = \check{C}^\bullet \otimes_R (M)$. Then $H_m^i(M) = H^i(\check{C}^\bullet(M))$.

Definition 2.3. Let $F \subseteq [n]$. We define \check{C}_F^\bullet to be the subcomplex of \check{C}^\bullet obtained by setting

$$\check{C}_F^i = \begin{cases} 0, & \text{if } i < |F|, \\ \bigoplus_{\substack{F \subseteq \Lambda \subseteq [n] \\ |\Lambda|=i}} R_{x_\Lambda}, & \text{otherwise.} \end{cases}$$

Lemma 2.4. Let I be a monomial ideal. Let $F \subseteq [n]$ and $\mathbf{a} \in \mathbb{Z}^n$ be such that $\text{Supp}(\mathbf{a}^-) = F$. Then $H_m^i(R/I)_{\mathbf{a}} = H^i(\check{C}_F^\bullet \otimes_R (R/I))_{\mathbf{a}}$.

Proof. This argument is used implicitly in the proof of [Tak05, Theorem 1]. Since $H_m^i(R/I)_{\mathbf{a}} = H^i(\check{C}^\bullet(R/I)_{\mathbf{a}})$, it suffices to show that $(\check{C}^\bullet(R/I))_{\mathbf{a}} = (\check{C}_F^\bullet \otimes_R (R/I))_{\mathbf{a}}$. This, in turn, stems from the fact that for all $1 \leq j \leq n$, $\check{C}_F^j \otimes_R (R/I)$ consists precisely of the direct summands of $\check{C}^j(R/I)$ that are non-zero in the multidegree \mathbf{a} . \square

3. PROOF OF THE MAIN THEOREM

Lemma 3.1. Let $I \subset R$ be a monomial ideal. Let $\mathbf{a} \in \mathbb{Z}^n$ and $j \in \text{Supp}(\mathbf{a}^+)$. Then the multiplication map

$$x_j : \text{Ext}_R^i(R/I, \omega_R)_{\mathbf{a}} \longrightarrow \text{Ext}_R^i(R/I, \omega_R)_{\mathbf{a}+\mathbf{e}_j}$$

is bijective.

Proof. We first claim that the multiplication map

$$x_j : H_m^{n-i}(R/I)_{-\mathbf{a}-\mathbf{e}_j} \longrightarrow H_m^{n-i}(R/I)_{-\mathbf{a}}$$

is bijective. By local duality [BH93, Theorem 3.6.19], this map is the Matlis dual of the multiplication by x_j on $\text{Ext}_R^i(R/I, \omega_R)_{\mathbf{a}}$; hence, it suffices to prove the claim.

Set $F = \text{Supp}(\mathbf{a}^+)$. Note that $\text{Supp}(\mathbf{a}^+ + \mathbf{e}_j) = F$. For all i , x_j acts as a unit on \check{C}_F^i . Therefore the homomorphism of complexes $\check{C}_F^\bullet \otimes_R (R/I) \rightarrow \check{C}_F^\bullet \otimes_R (R/I)$ induced by the multiplication map $x_j : \check{C}_F^i \otimes_R (R/I) \rightarrow \check{C}_F^i \otimes_R (R/I)$ is an isomorphism. The claim now follows from Lemma 2.4, which implies that $H_m^i(R/I)_{-\mathbf{a}-\mathbf{e}_j} = H^i(\check{C}_F^\bullet \otimes_R (R/I))_{-\mathbf{a}-\mathbf{e}_j}$ and $H_m^i(R/I)_{-\mathbf{a}} = H^i(\check{C}_F^\bullet \otimes_R (R/I))_{-\mathbf{a}}$. \square

The above lemma says that if I is a monomial ideal then $\text{Ext}_R^i(R/I, \omega_R)$ is a $(1, 1, \dots, 1)$ -determined module, in the sense of [Mil00, Definition 2.1].

Proof of Theorem 1.1. For $F \subseteq [n]$, let \mathcal{M}_F^i be a multigraded \mathbb{k} -basis for

$$\bigoplus_{\substack{\mathbf{a} \in \mathbb{N}^n \\ \text{Supp}(\mathbf{a}) \cap F = \emptyset}} \text{Ext}_R^i(R/I, \omega_R)_{\mathbf{e}_F - \mathbf{a}}.$$

Let $\mathcal{S}_i = \{(m, F) : F \subseteq [n] \text{ and } m \in \mathcal{M}_F^i\}$. Then it follows from Lemma 3.1 that \mathcal{S}_i is a Stanley decomposition of $\text{Ext}_R^i(R/I, \omega_R)$. In particular,

$$\dim \text{Ext}_R^i(R/I, \omega_R) = \max\{|F| : \mathcal{M}_F^i \neq \emptyset\}.$$

By the construction of \mathcal{M}_F^i , this Stanley decomposition satisfies the assumption of Proposition 2.2. Therefore

$$\begin{aligned} \text{reg}(\text{Ext}_R^i(R/I, \omega_R)) &\leq \max_{F \subseteq [n]} \{\max\{\|\deg m\| : m \in \mathcal{M}_F^i\}\} \\ &\leq \max_{F \subseteq [n]} \{|F| : \mathcal{M}_F^i \neq \emptyset\} \\ &= \dim \text{Ext}_R^i(R/I, \omega_R), \end{aligned}$$

as desired. (The second inequality follows from the fact that, for any $u \in \mathcal{M}_F^i$, one has $\|\deg u\| = |F| - \|(\deg u)^-\|$.) \square

We remark that, using [Tak05, Theorem 1] and local duality, one can determine whether $\mathcal{M}_F^i \neq \emptyset$ from certain subcomplexes of the Stanley-Reisner complex of the radical \sqrt{I} of I .

Acknowledgments. The authors thank B. Ulrich for helpful comments. This paper was written when the second author was visiting Purdue University in September 2009. He would like to thank his host, G. Caviglia, for his hospitality.

REFERENCES

- [BH93] W. Bruns and J. Herzog, *Cohen-Macaulay rings*, Cambridge Studies in Advanced Mathematics, vol. 39, Cambridge University Press, Cambridge, 1993. MR95h:13020 [↑1, 4](#)
- [CD03] M. Chardin and C. D'Cruz, *Castelnuovo-Mumford regularity: examples of curves and surfaces*, J. Algebra **270** (2003), no. 1, 347–360. MR2016666 (2004m:13036) [↑2](#)
- [CHH09] M. Chardin, D. T. Ha, and L. T. Hoa, *Castelnuovo-Mumford regularity of ext modules and homological degree*, 2009. arXiv:0903.4535v1 [math.AC]. [↑1](#)
- [Eis95] D. Eisenbud, *Commutative algebra*, Graduate Texts in Mathematics, vol. 150, Springer-Verlag, New York, 1995. With a view toward algebraic geometry. MR97a:13001 [↑3](#)
- [HH06] L. T. Hoa and E. Hyry, *Castelnuovo-Mumford regularity of canonical and deficiency modules*, J. Algebra **305** (2006), no. 2, 877–900. MR2266858 (2007g:13023) [↑1, 2](#)
- [Mil00] E. Miller, *The Alexander duality functors and local duality with monomial support*, J. Algebra **231** (2000), no. 1, 180–234. MR1779598 (2001k:13028) [↑4](#)
- [MS05] D. Maclagan and G. G. Smith, *Uniform bounds on multigraded regularity*, J. Algebraic Geom. **14** (2005), no. 1, 137–164. MR2092129 (2005g:14098) [↑2, 3](#)
- [Tak05] Y. Takayama, *Combinatorial characterizations of generalized Cohen-Macaulay monomial ideals*, Bull. Math. Soc. Sci. Math. Roumanie (N.S.) **48(96)** (2005), no. 3, 327–344. MR2165349 (2006e:13017) [↑1, 4, 5](#)
- [Yan00] K. Yanagawa, *Alexander duality for Stanley-Reisner rings and squarefree \mathbb{N}^n -graded modules*, J. Algebra **225** (2000), no. 2, 630–645. MR1741555 (2000m:13036) [↑1](#)

DEPARTMENT OF MATHEMATICS, PURDUE UNIVERSITY, WEST LAFAYETTE, IN 47901 USA.
E-mail address: nkummini@math.purdue.edu

DEPARTMENT OF MATHEMATICAL SCIENCE, FACULTY OF SCIENCE, YAMAGUCHI UNIVERSITY,
1677-1 YOSHIDA, YAMAGUCHI 753-8512, JAPAN.